GENERALIZED COHOMOLOGY OF PRO-SPECTRA

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ABSTRACT. We present a closed model structure for the category of pro-spectra in which the weak equivalences are detected by stable homotopy pro-groups. With some bounded-below assumptions, weak equivalences are also detected by cohomology as in the classical Whitehead theorem for spectra. We establish an Atiyah-Hirzebruch spectral sequence in this context, which makes possible the computation of topological K-theory (and other generalized cohomology theories) of pro-spectra.

1. INTRODUCTION

Ordinary singular cohomology of pro-objects is a useful tool in various mathematical concepts. For example, the cohomology of pro-spaces comes into play in the Bousfield-Kan viewpoint on R-completions of spaces [BK] [D] [I4]. Also, the singular cohomology with locally constant coefficients of the étale homotopy type of a scheme [AM] [F2] is isomorphic to the étale cohomology of the scheme. The continuous cohomology of a pro-finite group [S] is also an example of the same kind.

The notion of the cohomology of a pro-object is easy to describe. For any cofiltered system X, we define

$$(1.1) \qquad H^*(X) = \operatorname*{colim}_s H^*(X_s).$$

Wherever singular cohomology is useful, it is a good bet that generalized cohomology theories are also useful. This paper develops the foundations and tools necessary for studying these cohomology theories on pro-objects. In particular, we give a definition of generalized cohomology theories such that there is an Atiyah-Hirzebruch spectral sequence whose convergence is reasonably well-behaved (see Theorem 10.7).

We are only aware of one example of generalized cohomology theories applied to pro-objects: étale K-theory, which is the topological K-theory of the étale homotopy type of a scheme [F1]. In future work, we plan to use the foundations in this paper to develop a homotopy fixed points spectral sequence for pro-finite groups [FI]. We also intend to use étale BP and étale tmf, which are analogous to étale K-theory, in order to prove some results concerning quadratic forms over fields of characteristic p [DI1] [DI2].

Unfortunately, generalized cohomology theories are not as easy to define as ordinary cohomology. In view of Formula (1.1), the most obvious definition of the

topological K-theory of a pro-object is the formula

$$KU^*(X) = \operatorname*{colim}_s KU^*(X_s).$$

However, this turns out to be wrong computationally. Since there is an Atiyah-Hirzebruch spectral sequence whose abutment is each $KU^*(X_s)$ and since filtered colimits are exact, one might hope that the filtered colimit of these spectral sequences gives a computational tool for understanding $KU^*(X)$ as defined above. Unfortunately, the convergence of this colimit spectral sequence is terrible and therefore of no practical use. See Section 2 for a specific computational example of the problem.

The correct way to define generalized cohomology theories is more complicated. First, we define a closed model structure on the category of pro-spectra (see Section 5). This gives a homotopy theory for pro-spectra and in particular a homotopy category. Then, for any spectrum E, we simply define $E^r(X)$ to be the set of homotopy classes of maps from $\Sigma^{-r}X$ to the spectrum E considered as a constant pro-spectrum. In the same way, we can define a cohomology theory represented by any pro-spectrum.

Roughly speaking, the weak equivalences of the model structure on pro-spectra are defined in terms of pro-homotopy groups (see Section 8), and the fibrant pro-spectra are cofiltered diagrams of spectra whose homotopy groups are bounded above (see Section 6). This characterization of the fibrant pro-spectra connects with the fact that the Postnikov tower of the K-theory spectrum KU is key to the definition of étale K-theory [DF]. This paper gives a concrete explanation for why the Postnikov tower enters the picture because the Postnikov tower of KU, thought of as a pro-spectrum, is a fibrant replacement for KU.

The model structure for pro-spectra is analogous to a model structure for pro-spaces [I1]. Many of the technical complications of [I1] are questions of choosing basepoints and studying π_1-actions. These issues do not arise for pro-spectra, so many aspects of the theory for pro-spectra are easier than for pro-spaces.

There are at least two other reasonable ways to put a model structure on the category of pro-spectra [EH] [CI]. Each gives a distinct homotopy category, and each has particular uses. We have two strong pieces of evidence that the model structure under investigation in this paper is the correct one for the study of generalized cohomology theories of pro-spectra.

First, there is a Whitehead theorem showing that weak equivalences of pro-spectra can be detected by cohomology isomorphisms under some bounded below hypotheses (see Theorem 9.4). This is relevant in many applications because often pro-objects are constructed precisely for their cohomological properties.

The second piece of evidence is the Atiyah-Hirzebruch spectral sequence discussed above whose convergence is well-behaved.

1.1. **Organization.** The paper is organized as follows. We start with a concrete example in Section 2 to demonstrate what is wrong with the naive definition of topological K-theory of a pro-spectrum.

Section 3 is a review of the machinery of pro-categories and the very general "strict" model structure [EH, § 3.3] [I2]. The strict model structure is the starting point for all known homotopy theories of pro-categories.

Then we recall in Section 4 some ideas about spectra and stable homotopy theory. All of the basic properties of spectra that we will need are satisfied by all of the usual models of spectra, such as Bousfield-Friedlander spectra [BF], symmetric spectra

[HSS], or S-modules [EKMM]. Thus, the results in this paper can be viewed as applying to any of these categories of spectra.

Then in Section 5 we define the cofibrations, fibrations, and weak equivalences of pro-spectra, and we prove that they are a model structure. We assume that the reader has a basic familiarity with the terminology and standard results of model categories. The original source is [Q], but we follow the notation and terminology of [Hi] as closely as possible. Other references include [Ho] and [DS].

The next few sections contain useful properties of the homotopy theory of pro-spectra. If one is ever going to use the model structure on pro-spectra for anything, it is essential to know what the cofibrant and fibrant objects are. The cofibrant objects are easy to describe. In Section 6, we identify explicitly the fibrant pro-spectra.

In Section 7 we collect some results about computing homotopy classes of maps of pro-spectra in terms of homotopy classes of maps of spectra.

The actual definition of weak equivalences (see Definition 5.1) has the advantage that it is useful for proving model structure axioms. However, it lacks a computational aspect. In Section 8, we make precise the relationship between pro-homotopy groups and weak equivalences of pro-spectra. This tends to be useful in applications. In Section 9, we give yet another description of the weak equivalences in terms of cohomology.

Finally, in Section 10 we construct the Atiyah-Hirzebruch spectral sequence and prove that it is conditionally convergent in the sense of [B] for a large class of pro-spectra of interest. For example, the pro-spectra $\mathbb{R}P^\infty_{-\infty}$ and $\mathbb{C}P^\infty_{-\infty}$ [L] both belong to this class.

There are two ways to construct the Atiyah-Hirzebruch spectral sequence in ordinary stable homotopy theory converging to $[X, Y]^*$. One uses the skeletal filtration of X, and the other uses the Postnikov tower of Y [GM, App. B]. We use the approach with Postnikov towers here. This is no surprise because the Postnikov towers play such an important role in the model structure for pro-spectra.

We do not address the question of multiplicative properties of the Atiyah-Hirzebruch spectral sequence for pro-spectra, but we strongly suspect that everything works as expected.

2. WHAT IS THE K-THEORY OF A PRO-SPECTRUM?

We take the position that however the K-theory of a pro-spectrum is defined, there ought to be an Atiyah-Hirzebruch spectral sequence with reasonably good convergence properties. If K-theory is to be useful computationally, this is an appropriate expectation.

Let us assume for the moment that the K-theory of a pro-spectrum $X = \{X_s\}$ is defined to be $\operatorname{colim}_s KU^*(X_s)$. We will show that this definition does lead to an Atiyah-Hirzebruch spectral sequence, but the convergence is not at all good.

For each s, there is an Atiyah-Hirzebruch spectral sequence

$$H^p(X_s; \pi_q KU) \Rightarrow KU^{p+q}(X_s).$$

Since filtered colimits are exact, we can take colimits and obtain a spectral sequence

$$\operatorname*{colim}_s H^p(X_s; \pi_q KU) \Rightarrow \operatorname*{colim}_s KU^{p+q}(X_s).$$

The left side is just the ordinary cohomology $H^p(X; \pi_q KU)$ of X with coefficients in $\pi_q KU$, so this appears to be an Atiyah-Hirzebruch spectral sequence.

It remains to ask what kind of convergence properties this spectral sequence has. In the following particular case, we will show that the convergence is terrible; it's so bad that the E_2-term basically gives no information at all about the abutment.

For each $n \geq 0$, let X_n be the spectrum $\vee_{k=n}^{\infty} S^{2k}$, and let $X_{n+1} \to X_n$ be the obvious inclusion. Thus X is a pro-spectrum (in fact, a countable tower).

Since X_n is $(2n-1)$-connected, $H^p(X_n; \mathbb{Z})$ is zero for sufficiently large n. Therefore, $\mathrm{colim}_n H^p(X_n; \pi_q KU)$ is zero for all p and q. This means that the E_2-term of the above spectral sequence is zero.

On the other hand, $KU^{p+q}(X_n)$ is equal to $\prod_{k=n}^{\infty} \mathbb{Z}$ when $p+q$ is even and equal to zero when $p+q$ is odd. Therefore, when $p+q$ is even, $\mathrm{colim}_n KU^{p+q}(X_n)$ is equal to a quotient of $\prod_{k=1}^{\infty} \mathbb{Z}$, where two infinite sequences (a_k) and (b_k) are identified if a_k and b_k are different for only finitely many values of k. Another way to think of this group is the "germs at infinity" of functions $\mathbb{N} \to \mathbb{Z}$.

When $p+q$ is even, $\mathrm{colim}_n KU^{p+q}(X_n)$ is uncountable. Recall that the E_2-term of the spectral sequence was zero. The conclusion is that the above spectral sequence has disastrously bad convergence properties.

We are led to the conclusion that $\mathrm{colim}_s KU^*(X_s)$ is the wrong definition of the K-theory of a pro-spectrum. The point of the rest of this paper is to construct a suitable homotopy theory of pro-spectra such that $[X, KU]_{\mathrm{pro}}$ does have the desired computational properties. Here, KU means the constant pro-spectrum with value KU. When we define the weak equivalences in this homotopy theory later, it will be clear that for X in the previous paragraphs, the map $* \to X$ is a weak equivalence. Therefore, $[X, KU]_{\mathrm{pro}}$ is necessarily zero, which agrees with the computation of the E_2-term of the spectral sequence.

3. Preliminaries on Pro-Categories

We begin with a brief review of pro-categories. This section contains mostly standard material on pro-categories [SGA] [AM] [EH]. We conform to the notation and terminology of [I2].

3.1. **Pro-Categories.**

Definition 3.1. For a category $\mathcal{C}$, the category **pro-$\mathcal{C}$** has objects all cofiltering diagrams in $\mathcal{C}$, and

$$\mathrm{Hom}_{\mathrm{pro}\text{-}\mathcal{C}}(X, Y) = \lim_s \mathrm{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s).$$

Composition is defined in the natural way.

A **constant** pro-object is one indexed by the category with one object and one (identity) map. Let $\mathbf{c} : \mathcal{C} \to \mathrm{pro}\text{-}\mathcal{C}$ be the functor taking an object X to the constant pro-object with value X. Note that this functor makes $\mathcal{C}$ a full subcategory of pro-$\mathcal{C}$. The limit functor $\lim : \mathrm{pro}\text{-}\mathcal{C} \to \mathcal{C}$ is the right adjoint of c.

3.2. **Level representations.** A **level map** $X \to Y$ is a pro-map that is given by a natural transformation (so X and Y must have the same indexing category); this is very special kind of pro-map. A **level representation** of a pro-map $f : X \to Y$ is another pro-map $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\tilde{f}$ is a level map. Moreover, we require that

there are isomorphisms $X \to \tilde{X}$ and $Y \to \tilde{Y}$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
$$

of pro-maps commutes. Every map has a level representation [AM, App. 3.2]. See [C, App.] for a functorial construction of level representations.

A pro-object X satisfies a certain property **levelwise** if each X_s satisfies that property, and X satisfies this property **essentially levelwise** if it is isomorphic to another pro-object satisfying this property levelwise. Similarly, a level map $X \to Y$ satisfies a certain property **levelwise** if each $X_s \to Y_s$ has this property. A map of pro-objects satisfies this property **essentially levelwise** if it has a level representation satisfying this property levelwise.

The following purely technical lemma will be needed later in Lemma 5.14.

Lemma 3.2. *Let Y be a pro-object. Suppose that for some of the maps $t \to s$ in the indexing diagram for Y, there exists an object Z_{ts} and a factorization $Y_t \to Z_{ts} \to Y_s$ of the structure map $Y_t \to Y_s$. Also suppose that for every s, there exists at least one $t \to s$ with this property. The objects Z_{ts} assemble into a pro-object Z that is isomorphic to Y.*

Proof. We may assume that Y is indexed by a directed set I (in the sense that there is at most one map between any two objects of I) because every pro-object is isomorphic to a pro-object indexed by a directed set [EH, Thm. 2.1.6]. Define a new directed set K as follows. The elements of K consist of pairs (t, s) of elements of I such that $t \geq s$ and a factorization $Y_t \to Z_{ts} \to Y_s$ exists. If (t, s) and (t', s') are two elements of K, we say that $(t', s') \geq (t, s)$ if $s' \geq t$. It can easily be checked that this makes K into a directed set.

Note that the function $K \to I : (t, s) \mapsto s$ is cofinal in the sense of [AM, App. 1]. This means that we may reindex Y along this functor and assume that Y is indexed by K; thus we write $Y_{(t,s)} = Y_s$.

We define the pro-spectrum Z to be indexed by K by setting $Z_{(t,s)} = Z_{ts}$. If $(t', s') \geq (t, s)$, then the structure map $Z_{(t',s')} \to Z_{(t,s)}$ is the composition

$$
Z_{t's'} \to Y_{s'} \to Y_t \to Z_{ts},
$$

It can easily be checked that this gives a functor defined on K; here is where we use that the composition $Y_t \to Z_{ts} \to Y_s$ equals $Y_t \to Y_s$.

Finally, we must show that Z is isomorphic to Y. We use the criterion from [I1, Lem. 2.3] for detecting pro-isomorphisms. Given any (t, s) in K, choose u such that (u, t) is in K. Then there exists a diagram

$$
\begin{array}{ccc}
Z_{(u,t)} & \longrightarrow & Y_{(u,t)} = Y_t \\
\downarrow & \swarrow & \downarrow \\
Z_{(t,s)} & \longrightarrow & Y_{(t,s)} = Y_s.
\end{array}
$$

$\square$

3.3. **Strict Model Structures.** We make some remarks on the **strict model structure** for pro-categories, originally developed in [EH] and studied further in [I2]. The niceness hypothesis of [EH, § 2.3] is not satisfied by the categories of spectra that we will use, so the generalizations of [I2] really are necessary. The categories of pro-simplicial sets, pro-topological spaces, and any of the standard models for pro-spectra (such as Bousfield-Friedlander spectra [BF], symmetric spectra [HSS], or S-modules [EKMM]) all have strict model structures.

Let $\mathcal{C}$ be a proper model category. The **strict weak equivalences** of pro-$\mathcal{C}$ are the essentially levelwise weak equivalences (see Section 3.2). The **cofibrations** of pro-$\mathcal{C}$ are the essentially levelwise cofibrations. Finally, the **strict fibrations** of pro-$\mathcal{C}$ are maps that have the right lifting property with respect to the strict acyclic cofibrations. We use no adjective to describe the cofibrations because the cofibrations are the same in all known model structures on pro-categories.

The following theorem is the main result of [I2].

Theorem 3.3. *Let $\mathcal{C}$ be a proper model category. Then the classes of cofibrations, strict weak equivalences, and strict fibrations define a proper model structure on* pro-$\mathcal{C}$. *If $\mathcal{C}$ is simplicial, then this structure is also simplicial.*

For any two objects X and Y of pro-$\mathcal{C}$, the mapping space $\mathrm{Map}(X, Y)$ is equal to $\lim_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$ when $\mathcal{C}$ is simplicial.

We will need the following fact in a few places. It makes computations of mapping spaces significantly easier.

Proposition 3.4. *Let $\mathcal{C}$ be a proper simplicial model category. Let X be a cofibrant object of* pro-$\mathcal{C}$. *Let Y be any levelwise fibrant object of* pro-$\mathcal{C}$ *with strict fibrant replacement $\hat{Y}$. Then the homotopically correct mapping space $\mathrm{Map}(X, \hat{Y})$ is weakly equivalent to* $\mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$.

Proof. Since $\mathrm{Map}(X, \hat{Y})$ is homotopically correct, it doesn't matter which strict fibrant replacement $\hat{Y}$ that we consider. Therefore, we may choose one with particularly good properties. Use the method of [I2, Lem. 4.7] to factor the map $Y \to *$ into a strict acyclic cofibration $Y \to \hat{Y}$ followed by a strict fibration $\hat{Y} \to *$. This particular construction gives that $Y \to \hat{Y}$ is a levelwise weak equivalence and that $\hat{Y}$ is levelwise fibrant.

Since X is cofibrant and $\hat{Y}$ is strict fibrant, the pro-space $s \mapsto \mathrm{colim}_t \mathrm{Map}(X_t, \hat{Y}_s)$ is also strict fibrant. This can be seen by inspecting the explicit description of strict fibrations given in [I2, Defn. 4.2]. Therefore, $\mathrm{Map}(X, \hat{Y}) = \lim_s \mathrm{colim}_t \mathrm{Map}(X_t, \hat{Y}_s)$ is weakly equivalent to $\mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, \hat{Y}_s)$ because homotopy limit is the derived functor of limit with respect to the strict model structure [EH, Rem. 4.2.11].

The map $\mathrm{colim}_t \mathrm{Map}(X_t, Y_s) \to \mathrm{colim}_t \mathrm{Map}(X_t, \hat{Y}_s)$ is a weak equivalence because $Y_s \to \hat{Y}_s$ is a weak equivalence between fibrant objects. Homotopy limits preserve levelwise weak equivalences, so the map

$$\mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s) \to \mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, \hat{Y}_s)$$

is a weak equivalence.   □

We will next show that construction of the strict model structure respects Quillen equivalences [Hi, Defn. 8.5.20]. It was an oversight that this result was not included in [I2].

If $F : \mathcal{C} \to \mathcal{D}$ is any functor, then there is another functor $F : \text{pro-}\mathcal{C} \to \text{pro-}\mathcal{D}$ defined by applying F levelwise to any object in pro-$\mathcal{C}$. If $G : \mathcal{D} \to \mathcal{C}$ is right adjoint to F, then G is also right adjoint to F on pro-categories.

Theorem 3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories such that the strict model structures on pro-$\mathcal{C}$ and pro-$\mathcal{D}$ exist (for example, if $\mathcal{C}$ and $\mathcal{D}$ are proper). If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are a Quillen adjoint pair, then F and G are also a Quillen adjoint pair between pro-$\mathcal{C}$ and pro-$\mathcal{D}$ equipped with their strict model structures. If F and G are a Quillen equivalence between $\mathcal{C}$ and $\mathcal{D}$, then they are also a Quillen equivalence between pro-$\mathcal{C}$ and pro-$\mathcal{D}$.*

Proof. First suppose that F and G are a Quillen adjoint pair on $\mathcal{C}$ and $\mathcal{D}$. Since F takes cofibrations in $\mathcal{C}$ to cofibrations in $\mathcal{D}$, it takes levelwise cofibrations in pro-$\mathcal{C}$ to levelwise cofibrations in pro-$\mathcal{D}$. Thus, F preserves essentially levelwise cofibrations.

Since F takes acyclic cofibrations in $\mathcal{C}$ to acyclic cofibrations in $\mathcal{D}$, it similarly preserves essentially levelwise acyclic cofibrations. However, the essentially levelwise acyclic cofibrations are the same as the strict acyclic cofibrations [I2, Prop. 4.11]. This shows that F preserves strict acyclic cofibrations. Thus F and G are a Quillen adjoint pair.

Now suppose that F and G are a Quillen equivalence on $\mathcal{C}$ and $\mathcal{D}$. To show that F and G are a Quillen equivalence on pro-$\mathcal{C}$ and pro-$\mathcal{D}$, let X be a cofibrant object of pro-$\mathcal{C}$ and let Y be a strict fibrant object of pro-$\mathcal{D}$. Suppose that $g : X \to GY$ is a strict weak equivalence; we want to show that its adjoint $f : FX \to Y$ is also a strict weak equivalence.

We may assume that X is levelwise cofibrant. By [I2, Lem. 4.5], we may also assume that Y is levelwise fibrant. Using the level replacement of [AM, App. 3.2], we may reindex X and Y in such a way that X is still levelwise cofibrant, Y is still levelwise fibrant, and $g : X \to GY$ is a level map. However, we are not allowed to assume that g is a levelwise weak equivalence because this may require a different reindexing.

Use the method of [I2, Lem. 4.6] to factor f into a strict cofibration $i : X \to Z$ followed by a strict acyclic fibration $p : Z \to GY$. This particular construction gives that i is a levelwise cofibration and that p is a levelwise acyclic fibration. In particular, this implies that Z is levelwise cofibrant since X is. The two-out-of-three axiom implies that i is a strict acyclic cofibration, even though it is not a levelwise acyclic cofibration.

The adjoint $p' : FZ \to Y$ of p is a levelwise weak equivalence because F and G are a Quillen equivalence between $\mathcal{C}$ and $\mathcal{D}$. This works because Z is levelwise cofibrant, Y is levelwise fibrant, and p is a level map. The map $Fi : FX \to FZ$ is a strict acyclic cofibration because left Quillen functors preserve acyclic cofibrations. The map f is the composition of Fi with p', so f is a strict weak equivalence.

Now assume that $f : FX \to Y$ is a strict weak equivalence. To show that its adjoint $g : X \to GY$ is also a strict weak equivalence, use the dual argument. □

In particular, $- \wedge S^1$ and $\text{Map}(S^1, -)$ are adjoint functors on spectra. Thus, they induce adjoint functors on pro-spectra also. Similarly to spectra, we define the suspension functor Σ and loops functor Ω on pro-spectra as the derived functors of these functors. We won't recall the basic details of spectra until the next section. For now, it is enough to know that the functors $- \wedge S^1$ and $\text{Map}(S^1, -)$ are a Quillen equivalence from the category of spectra to itself.

Theorem 3.6. *The functors* $- \wedge S^1$ *and* $\mathrm{Map}(S^1, -)$ *are a Quillen equivalence from the strict model structure on pro-spectra to itself.*

In other words, the strict model structure on pro-spectra is stable in the sense of [Ho].

Proof. This is an immediate application of Theorem 3.5.   □

4. Preliminaries on Spectra

This section contains some results on spectra and stable homotopy theory. Much of the material is well-known.

We work with a proper simplicial model structure on a category of spectra such as Bousfield-Friedlander spectra [BF], S-modules [EKMM], or symmetric spectra [HSS]. We assume that the model structure is cofibrantly generated. Moreover, the cofiber of any generating cofibration must be a sphere. If the dimension of the sphere is k, then we call such a map a **generating cofibration of dimension k**.

We also need that stable weak equivalences are detected by stable homotopy groups (this is true even for symmetric spectra if the stable homotopy groups are properly defined). Also, stable homotopy groups commute with colimits along transfinite compositions of cofibrations.

As usual, the symbol Σ refers to the **suspension** functor, the left derived version of the functor $- \wedge S^1$. Thus ΣX is defined to be $\tilde{X} \wedge S^1$ for a cofibrant replacement $\tilde{X}$ of X. Similarly, the symbol Ω refers to the **loops** functor, the right derived version of the functor $\mathrm{Map}(S^1, -)$. This means that ΩX is defined to be $\mathrm{Map}(S^1, \hat{X})$ for a fibrant replacement $\hat{X}$ of X. The key property of Σ and Ω is that they are inverse equivalences on the stable homotopy category.

Let $[X, Y]$ be the set of stable homotopy classes from X to Y, and let $[X, Y]^r$ be the set of stable homotopy classes of degree r from X to Y. If the functor Σ^r is defined to be Ω^{-r} for $r \leq 0$, then $[X, Y]^r$ is equal to

$$[\Sigma^{-r} X, Y] = [X, \Sigma^r Y]$$

for all r.

An **Eilenberg-Mac Lane spectrum** is a spectrum whose stable homotopy groups are zero except in one dimension.

4.1. **n-Equivalences.** In the next two subsections, we study some special kinds of maps of spectra that play a central role in the model structure for pro-spectra.

Definition 4.1. A map f of spectra is an **n-equivalence** if $\pi_k f$ is an isomorphism for $k < n$ and $\pi_n f$ is a surjection. A map f is a **co-n-equivalence** if $\pi_k f$ is an isomorphism for $k > n$ and $\pi_n f$ is an injection.

Definition 4.2. A spectrum X is **bounded below** if the map $* \to X$ is an n-equivalence for some n. A spectrum X is **bounded above** if the map $X \to *$ is a co-n-equivalence for some n.

Of course, a bounded below spectrum is a spectrum whose homotopy groups vanish below some (arbitrarily small) dimension, and a bounded above spectrum is a spectrum whose homotopy groups vanish above some (arbitrarily large) dimension.

Lemma 4.3. *A map is an n-equivalence if and only if its homotopy cofiber C satisfies $\pi_k C = 0$ for all $k \leq n$. A map is a co-n-equivalence if and only if its homotopy fiber F satisfies $\pi_k F = 0$ for all $k \geq n$.*

Proof. This follows immediately from the long exact sequence of homotopy groups of a homotopy cofiber sequence or homotopy fiber sequence. □

Lemma 4.4. *Base changes along fibrations preserve n-equivalences and co-n-equivalences. Cobase changes along cofibrations preserve n-equivalences and co-n-equivalences.*

Proof. First consider a pullback square

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \scriptstyle g \downarrow & & \downarrow \scriptstyle f \\ X & \underset{p}{\longrightarrow} & Y \end{array}$$

in which p is a fibration and f is a co-n-equivalence. Let F be the homotopy fiber of f. By Lemma 4.3, $\pi_k F = 0$ for all $k \geq n$. The pullback square is a homotopy pullback square because p is a fibration, so the homotopy fiber of g is also F. By Lemma 4.3 again, g is a co-n-equivalence.

Now suppose that f is an n-equivalence. Then the homotopy cofiber C of f satisfies $\pi_k C = 0$ for all $k \leq n$. Note that C is the suspension ΣF of the homotopy fiber F. Since F is also the homotopy fiber of g, C is also the homotopy cofiber of g. Lemma 4.3 again tells us that g is an n-eqvuivalence.

The proof for cobase changes along cofibrations is dual. □

4.2. Co-n-Fibrations and n-Cofibrations.

Now we need some results on how the n-equivalences interact with the fibrations and cofibrations.

Definition 4.5. A map of spectra is a **co-n-fibration** if it has the right lifting property with respect to all generating acyclic cofibrations and all generating cofibrations of dimension greater than n. A map of spectra is an **n-cofibration** if it has the left lifting property with respect to all co-n-fibrations.

Note that co-n-fibrations and n-cofibrations are characterized by lifting properties with respect to each other. Also, the class of n-cofibrations is the same as the class of retracts of relative J_n-cell complexes, where $\boldsymbol{J_n}$ is the set of generating acyclic cofibrations together with the set of generating cofibrations of dimension greater than n [Hi, Cor. 10.5.23, Defn. 12.4.7].

When $n = -\infty$, the definitions reduce to the usual definitions of cofibrations and acyclic fibrations. When $n = \infty$, the definitions reduce to the usual definitions of acyclic cofibrations and fibrations.

Lemma 4.6. *Every acyclic fibration is a co-n-fibration, and every co-n-fibration is a fibration. Every acyclic cofibration is an n-cofibration, and every n-cofibration is a cofibration. If $m \geq n$, then every m-cofibration is an n-cofibration, and every co-n-fibration is a co-m-fibration.*

Proof. Compare the lifting properties given in Definition 4.5 to the usual lifting properties of cofibrations, acyclic cofibrations, fibrations, and acyclic fibrations. □

Lemma 4.7. *For any n, maps of spectra factor functorially into n-cofibrations followed by co-n-fibrations.*

Proof. Apply the small object argument [Hi, Prop. 10.5.16] to the set J_n of acyclic generating cofibrations together with generating cofibrations of dimension greater than n. □

When working with n-cofibrations and co-n-fibrations, we use the following two propositions frequently to pass between lifting properties and properties of homotopy groups as expressed in the notions of n-equivalences and co-n-equivalences.

Proposition 4.8. *A map of spectra is a co-n-fibration if and only if it is a fibration and a co-n-equivalence.*

Proof. This is proved in [CDI, Thm. 8.6]. Here is the basic idea. Obstructions for lifting generating cofibrations of dimension k with respect to a fibration p belong to the $(k-1)$st stable homotopy group of the fiber of p. This connects to co-n-equivalences via Lemma 4.3. $\square$

Proposition 4.9. *A map f of spectra is an n-cofibration if and only if it is a cofibration and an n-equivalence.*

Proof. Consider the class C of all maps that are cofibrations and n-equivalences. We will first show that C contains all retracts of J_n-cell complexes and thus contains all n-cofibrations.

Acyclic cofibrations belong to C, as do generating cofibrations of dimension greater than n. Therefore C contains J_n. An argument similar to the proof of Lemma 4.4 implies that C is closed under cobase changes. Next, observe that C is closed under transfinite compositions of cofibrations because stable homotopy groups commute with filtered colimits along such compositions. Finally, retracts preserve cofibrations and n-equivalences, so C is closed under retracts. This finishes one implication.

For the other implication, assume that f is a cofibration and n-equivalence. Let C be the cofiber of f. By Lemma 4.3, the desuspension ΩC has the property that $\pi_k \Omega C = 0$ for $k \leq n - 1$.

We have to show that f has the left lifting property with respect to any co-n-fibration p. By Proposition 4.8 and Lemma 4.3, the fiber F of p has the property that $\pi_k F = 0$ for $k \geq n$.
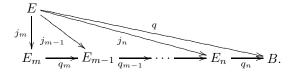
A lifting problem for f with respect to p has an obstruction belonging to $[\Omega C, F]$ [CDI, Cor. 8.4]. However, the conditions on the homotopy groups of ΩC and F guarantee that $[\Omega C, F]$ equals 0. Therefore, the obstruction to lifting must vanish, and the desired lift exists. $\square$

We will now show how to build co-n-fibrations out of fibrations whose fibers are Eilenberg-Mac Lane spectra.

Lemma 4.10. *Let m and n be any integers. Any co-m-fibration is a retract of a map that can be factored into a finite composition of co-n-fibrations and fibrations whose fibers are Eilenberg-Mac Lane spectra.*

Proof. If $n \geq m$, then any co-m-fibration is a co-n-fibration. Thus, we may assume that $m > n$.

Let $q : E \to B$ be a co-m-fibration. First use Lemma 4.7 to factor q into an n-cofibration $j_n : E \to E_n$ followed by a co-n-fibration $q_n : E_n \to B$. Repeat by factoring j_{k-1} into a k-cofibration $j_k : E \to E_k$ followed by a co-k-fibration $q_k : E_k \to E_{k-1}$. We obtain a diagram

Let $p : E_m \to B$ be the composition of the maps $q_n, \ldots, q_m$. Lifting the m-cofibration j_m with respect to the co-m-fibration q shows that q is a retract of p.

The map q_n is a co-n-fibration by construction. For $n + 1 \leq k \leq m$, let F_k be the fiber of q_k. Since q_k is a co-k-fibration, $\pi_r F_k = 0$ for $r \geq k$ by Lemma 4.3 and Proposition 4.8. Using that j_k is a k-equivalence, that j_{k-1} is a $(k-1)$-equivalence, and that $j_{k-1} = q_m j_m$, a small diagram chase verifies that q_k is a $(k-1)$-equivalence. This implies that $\pi_r F_k = 0$ for $r \leq k - 2$. Hence $\pi_r F_k$ can only be non-zero when $r = k - 1$, so F_k is an Eilenberg-Mac Lane spectrum. $\square$

4.3. **Mapping spaces and homotopy classes.** We next show that n-cofibrations interact appropriately with tensors. This will be needed to show that the model structure on pro-spectra is simplicial. If X is a spectrum and K is a simplicial set, recall that $X \otimes K$ is defined to be $X \wedge K_+$.

Proposition 4.11. *Suppose that $f : A \to B$ is an n-cofibration and $i : K \to L$ is a cofibration of simplicial sets. Then the map*

$$g : A \otimes L \amalg_{A \otimes K} B \otimes K \to B \otimes L$$

is also an n-cofibration.

Proof. The map i is a transfinite composition of cobase changes of maps of the form $\partial \Delta[j] \to \Delta[j]$. Therefore, the map g is a transfinite composition of cobase changes of maps of the form

$$A \otimes \Delta[j] \amalg_{A \otimes \partial \Delta[j]} B \otimes \partial \Delta[j] \to B \otimes \Delta[j].$$

Since n-cofibrations are defined by a left lifting property, n-cofibrations are preserved by cobase changes and transfinite compositions. Therefore, we may assume that i is the map $\partial \Delta[j] \to \Delta[j]$.

Spectra are a simplicial model category and f is a cofibration by Lemma 4.6, so g is also a cofibration. By Proposition 4.9, we need only show that g is an n-equivalence. Let C be the cofiber of f. Then the cofiber of g is $C \wedge S^j$, where the simplicial set S^j is the sphere $\Delta[j]/\partial \Delta[j]$ based at the image of $\partial \Delta[j]$. By Lemma 4.3, we need only show that $\pi_k(C \wedge S^j) = 0$ for all $k \leq n$. Since $\pi_k C = 0$ for all $k \leq n$ by Lemma 4.3 and $\pi_k(C \wedge S^j) = \pi_{k-j} C$, it follows that $\pi_k(C \wedge S^j) = 0$ for $k \leq j + n$. This suffices since $j \geq 0$. $\square$

Corollary 4.12. *Let $A \to B$ be an n-cofibration, and let $X \to Y$ be a co-n-fibration. The map*

$$f : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is an acyclic fibration of simplicial sets.

Proof. This follows from the lifting property characterization of acyclic fibrations, adjointness, and Proposition 4.11. $\square$

The next result is a highly technical lemma that will be needed in one place later.

Lemma 4.13. *Let $X \to Y$ be a map of spectra such that $* \to X$ is an $(n-1)$-equivalence and such that the map $\pi_n X \to \pi_n Y$ is zero. Let $Z \to *$ be a co-$(n+1)$-equivalence. Then the map $[Y, Z] \to [X, Z]$ is zero.*

Proof. We may assume that X, Y, and Z are both cofibrant and fibrant. For each element of $\pi_n X$, choose a representative $S^n \to X$. This gives a map $\vee S^n \to X$; let X' be its cofiber. Note that $\pi_n X'$ equals zero by construction. Also note that $\pi_k X'$

is isomorphic to $\pi_k X$ for $k < n$. The homotopy groups of X' vanish in dimensions less than or equal to n, and the homotopy groups of Z vanish in dimensions greater than or equal to $n+1$. This guarantees that $[X', Z]$ is zero. From the exact sequence

$$[X', Z] \to [X, Z] \to [\vee S^n, Z],$$

we see that $[X, Z] \to [\vee S^n, Z]$ is injective.

Now the composition $\vee S^n \to X \to Y$ is homotopy trivial by assumption. Therefore, the composition

$$[Y, Z] \to [X, Z] \to [\vee S^n, Z]$$

is zero. But the second map is injective, so we can conclude that the first map is zero. $\square$

5. Model Structure

We now define a model structure for pro-spectra.

Definition 5.1. A map of pro-spectra f is a $\boldsymbol{\pi_*}$**-weak equivalence** if f is an essentially levelwise n-equivalence for every n.

This means that for every n, f has a level representation that is a levelwise an n-equivalence. Beware that π_*-weak equivalences do not have to be strict weak equivalences. The point is that different level representations may be required for different values of n.

The terminology may appear strange at this point. In Section 8, we will show that the π_*-weak equivalences can be recharacterized in terms of of pro-homotopy groups.

Example 5.2. Recall the pro-spectrum X from Section 2, where X_n is the wedge $\vee_{k=n}^{\infty} S^{2k}$. We claimed in Section 2 that the map $* \to X$ is a π_*-weak equivalence. To see that $* \to X$ is an essentially levelwise m-equivalence, restrict X to the subdiagram X' consisting of those X_n with $2n > m$. The subdiagram X' is cofinal in X, so X and X' are isomorphic as pro-spectra. The level map from $*$ to X' is a levelwise m-equivalence.

The following lemma shows that our model structure is a localization of the strict model structure (see Section 3.3).

Lemma 5.3. *Strict weak equivalences are π_*-weak equivalences.*

Proof. A levelwise weak equivalence is also a levelwise n-equivalence. $\square$

Definition 5.4. A map of pro-spectra is a **cofibration** if it is an essentially levelwise cofibration.

Definition 5.5. A map of pro-spectra is a π_***-fibration** if it has the right lifting property with respect to all π_*-acyclic cofibrations.

The terminology here emphasizes that the notion of cofibration is the same in all known model structures for pro-spectra. On the other hand, the fibrations vary among the different model structures.

It requires some work to establish that these definitions give a model structure on the category of pro-spectra. We begin by collecting various technical lemmas. By the end of this section, we will be able to prove that the model structure exists.

5.1. **Two-out-of-Three Axiom.** This subsection deals with the two-out-of-three axiom for π_*-weak equivalences. Typically this axiom is automatic from the definition, but we have to do a little work.

Lemma 5.6. *Suppose that f and g are two composable morphisms of pro-spectra. If any two of f, g, and gf are essentially levelwise n-equivalences, then the third is an essentially levelwise $(n-1)$-equivalence.*

Proof. The proofs of [I2, Lem. 3.5] and [I2, Lem. 3.6], which concern the two-out-of-three axiom for essentially levelwise weak equivalences, can be applied. To make these proofs work, two formal properties of n-equivalences are required. First, n-equivalences are preserved by base changes along fibrations and cobase changes along cofibrations (see Lemma 4.4). Second, if $f : X \to Y$ and $g : Y \to Z$ are maps of ordinary spectra and any two of f, g, and gf are n-equivalences, then the third is an $(n-1)$-equivalence. $\square$

Proposition 5.7. *The π_*-weak equivalences of pro-spectra given in Definition 5.1 satisfy the two-out-of-three axiom.*

Proof. Let f and g be two composable maps of pro-spectra, and suppose that two of the maps f, g, and gf are π_*-weak equivalences. By Lemma 5.6, the third is an essentially levelwise $(n-1)$-equivalence for every n. $\square$

5.2. π_*-**Acyclic Cofibrations.** We shall find it useful to study the essentially levelwise n-cofibrations. Beware that we do not know (yet) that these maps are the same as maps that are both essentially levelwise cofibrations and essentially levelwise n-equivalences. The difficulty is that the reindexing required to replace a map by a levelwise cofibration may not agree with the reindexing required to replace the same map by a levelwise n-equivalence.

Lemma 5.8. *Any essentially levelwise n-equivalence factors into an essentially levelwise n-cofibration followed by a strict acyclic fibration.*

Proof. We may assume that f is a level map that is a levelwise n-equivalence. Use the method of [I2, Lem. 4.6] to factor f into a levelwise cofibration i followed by a strict acyclic fibration p. By [I2, Lem. 4.4], p is also a levelwise acyclic fibration.

For each s, we have $f_s = p_s i_s$. Since f_s is an n-equivalence and p_s is a weak equivalence, it follows that i_s is also an n-equivalence. Now use Proposition 4.9 to conclude that i is a levelwise n-cofibration. $\square$

Proposition 5.9. *A map is a cofibration and essentially levelwise n-equivalence if and only if it is an essentially levelwise n-cofibration.*

Proof. First suppose that i is an essentially levelwise n-cofibration. Then i is an essentially levelwise cofibration because every n-cofibration is a cofibration. Similarly, i is an essentially levelwise n-equivalence because every n-cofibration is an n-equivalence.

For the other direction, let i be a cofibration and essentially levelwise n-equivalence. By Lemma 5.8, i factors into an essentially levelwise n-cofibration j followed by a strict acyclic fibration p. Then p has the right lifting property with respect to the cofibration i because of the strict structure, so i is a retract of j. Essentially levelwise n-cofibrations are closed under retract by [I3, Cor. 5.6]. $\square$

5.3. π_*-**Fibrations.**

Lemma 5.10. *Every π_*-fibration is a strict fibration, and every strict acyclic fibration is a π_*-acyclic fibration.*

Proof. For the first claim, observe that Lemma 5.3 guarantees every strict acyclic cofibration is a π_*-acyclic cofibration. Now use the lifting property definitions of π_*-fibrations and strict fibrations.

For the second claim, recall that strict acyclic fibrations have the right lifting property with respect to all cofibrations and therefore with respect to π_*-acyclic cofibrations. This means that a strict acyclic fibration is a π_*-fibration. To show that it is also a π_*-weak equivalence, use Lemma 5.3.   □

Lemma 5.11. *Every π_*-fibration is an essentially levelwise fibration.*

Proof. This follows from Lemma 5.10 and the fact that strict fibrations are essentially levelwise fibrations [I2, Lem. 4.5].   □

Next we produce some examples of π_*-fibrations.

Lemma 5.12. *Let $X \to Y$ be a co-m-fibration for some m. Then the constant map $p : cX \to cY$ is a π_*-fibration.*

Proof. We show that p has the desired right lifting property. Let $i : A \to B$ be a π_*-acyclic cofibration, so i is an essentially levelwise m-equivalence. By Proposition 5.9, we may assume that i is a levelwise m-cofibration.

Suppose given a square

$$
\begin{array}{ccc}
A & \longrightarrow & cX \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & cY
\end{array}
$$

of pro-spectra. This square is represented by a square

$$
\begin{array}{ccc}
A_s & \longrightarrow & X \\
{\scriptstyle i_s}\downarrow & & \downarrow \\
B_s & \longrightarrow & Y
\end{array}
$$

of spectra for some s. Now i_s is an m-cofibration and $X \to Y$ is a co-m-fibration, so this last square has a lift. The lift represents the desired lift.   □

5.4. **Small object argument.** Eventually we will produce factorziations with a dual version of the generalized small object argument [C]. The next results are the technical details that allow us to apply this technique.

Definition 5.13. Given a level map $f : X \to Y$, let $\boldsymbol{F(f)}$ be the set of fibrations of spectra defined as follows. For each s and each n, consider the functorial factorization of $f_s : X_s \to Y_s$ into an n-cofibration $i_{s,n} : X_s \to Z_{s,n}$ followed by a co-n-fibration $p_{s,n} : Z_{s,n} \to Y_s$ as in Lemma 4.7. Let $F(f)$ be the set of all such maps $p_{s,n}$.

Lemma 5.14. *A map $i : A \to B$ is a π_*-acyclic cofibration if and only if it has the left lifting property with respect to all constant pro-maps $cX \to cY$ in which $X \to Y$ is a co-m-fibration for some m.*

Proof. One implication is shown in Lemma 5.12. For the other implication, suppose that i has the desired lifting property. Since acyclic fibrations of spectra are co-m-fibrations, i has the left lifting property with respect to all maps $cX \to cY$ in which $X \to Y$ is an acyclic fibration of spectra. By [I2, Prop. 5.5], this implies that i is a cofibration.

Fix an n. We show that i is an essentially levelwise n-equivalence. From the previous paragraph, we may assume that i is a levelwise cofibration.
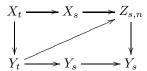
Consider the square

$$\begin{array}{ccc} X & \longrightarrow & cZ_{s,n} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & cY_s \end{array}$$

of pro-spectra, where the map $X \to cZ_{s,n}$ is the composition of the canonical map $X \to cX_s$ together with the map $i_{s,n} : X_s \to Z_{s,n}$ (see Definition 5.13) and the map $Y \to cY_s$ is the canonical map. Our assumption gives us a lift in this diagram because $Z_{s,n} \to Y_s$ is a co-n-fibration. This means that we have a diagram

$$\begin{array}{ccccc} X_t & \longrightarrow & X_s & \longrightarrow & Z_{s,n} \\ \downarrow & & & \nearrow & \downarrow \\ Y_t & \longrightarrow & Y_s & \longrightarrow & Y_s \end{array}$$

for some t, which can be rewritten as

$$\begin{array}{ccccc} X_t & \longrightarrow & X_s & \longrightarrow & X_s \\ \downarrow & & \downarrow & & \downarrow \\ Y_t & \longrightarrow & Z_{s,n} & \longrightarrow & Y_s. \end{array}$$

Finally, Lemma 3.2 shows that the objects $Z_{s,n}$ can be assembled into a pro-spectrum that is isomorphic to Y. Thus, the maps $X_s \to Z_{s,n}$ give a level representation of f. Each map $X_s \to Z_{s,n}$ is a levelwise n-equivalence, so $X \to Y$ is an essentially levelwise n-equivalence. $\square$

Lemma 5.15. *Consider a square*

$$\begin{array}{ccc} X & \longrightarrow & cE \\ f \downarrow & & \downarrow p \\ Y & \longrightarrow & cB \end{array}$$

of pro-spectra in which p is a constant pro-map such that $E \to B$ is a co-n-fibration for some n. This diagram factors as

$$\begin{array}{ccccc} X & \longrightarrow & cZ_{s,n} & \longrightarrow & cE \\ f \downarrow & & \downarrow & & \downarrow p \\ Y & \longrightarrow & cY_s & \longrightarrow & cB \end{array}$$

for some $p_{s,n} : Z_{s,n} \to Y$ belonging to $F(f)$.

Proof. We may assume that f is a level map. The original square is represented by a diagram

$$
\begin{array}{ccc}
X_s & \longrightarrow & E \\
\downarrow & & \downarrow \\
Y_s & \longrightarrow & B
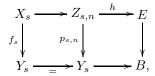\end{array}
$$

for some s. This gives us a square

$$
\begin{array}{ccc}
X_s & \longrightarrow & E \\
{\scriptstyle i_{s,n}} \downarrow & & \downarrow \\
Z_{s,n} & \longrightarrow & B
\end{array}
$$

in which the bottom horizontal map is the composition of $p_{s,n}$ with the given map $Y_s \to B$. Note that the left vertical map is an n-cofibration (see Definition 5.13) and the right vertical map is a co-n-fibration. Therefore, a lift h exists in this diagram.

Such a lift h gives us a diagram

$$
\begin{array}{ccccc}
X_s & \longrightarrow & Z_{s,n} & \overset{h}{\longrightarrow} & E \\
{\scriptstyle f_s} \downarrow & & {\scriptstyle p_{s,n}} \downarrow & & \downarrow \\
Y_s & \underset{=}{\longrightarrow} & Y_s & \longrightarrow & B,
\end{array}
$$

and this produces the desired factorization. $\qquad\square$

5.5. **The π_*-model structure.** We are now ready to prove that the model structure axioms are satisfied.

Theorem 5.16. *The cofibrations, π_*-weak equivalences, and π_*-fibrations are a simplicial proper model structure on the category of pro-spectra.*

We call this the **π_*-model structure** for pro-spectra.

Proof. The category of pro-spectra has all limits and colimits since the category of spectra does [I1, Prop. 11.1]. The two-out-of-three axiom for π_*-weak equivalences is not automatic; we proved this in Proposition 5.7. Retracts preserve essentially levelwise properties [I3, Cor. 5.6]. Therefore, retracts preserve cofibrations and π_*-weak equivalences. Retracts preserve π_*-fibrations because retracts preserve lifting properties.

See [I2, Lem. 4.6] for factorizations into cofibrations followed by maps that are strict acyclic fibrations. By Lemma 5.10, strict acyclic fibrations are π_*-acyclic fibrations. This gives factorizations into cofibrations followed by π_*-acyclic fibrations.

We next construct factorizations into π_*-acyclic cofibrations followed by π_*-fibrations. The generalized small object argument [C] can be applied to the class of maps $cX \to cY$ such that $X \to Y$ is a co-m-fibration for some m. Actually, we are applying the categorical dual. The cosmallness hypothesis is proved in [CI, Prop. 3.3]. The other hypothesis is Lemma 5.15.

We use Lemma 5.14 to conclude that the first map in the factorization is a π_*-acyclic cofibration. To conclude that the second map is a π_*-fibration, we use Lemma 5.12 and note that the second map is constructed as a composition of a transfinite tower of maps that are base changes of maps of the form $cX \to cY$ such that $X \to Y$

is a co-m-fibration for some m. Now apply the formal properties of right lifting properties.

One of the lifting axioms follows by definition. The other follows from the retract argument [Hi, Prop. 7.2.2]. In more detail, any π_*-acyclic fibration p can be factored into a cofibration i followed by a strict acyclic fibration p. Then i is a π_*-weak equivalence by the two-out-of-three axiom and the fact that strict weak equivalences are π_*-weak equivalences (see Lemma 5.3). Hence p has the right lifting property with respect to i, so p is a retract of q. It follows that p is a strict acyclic fibration, so it has the right lifting property with respect to all cofibrations.

The simplicial structure is analogous to the simplicial structure for pro-spaces [I1, § 16]. Beware that the definitions of tensor and cotensor are straightforward for finite simplicial sets but are slightly subtle in general. We need to show that if $i : K \to L$ is a cofibration of finite simplicial sets and $f : A \to B$ is a cofibration of pro-spectra, then the map

$$g : A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$$

is a cofibration of pro-spectra that is a π_*-weak equivalence if either i is an acyclic cofibration or f is a π_*-acyclic cofibration. The fact that g is a cofibration follows from the fact that the strict model structure is simplicial [I2, Thm. 4.16]. The case when i is an acyclic cofibration also follows from the strict structure.

It remains to assume that f is a π_*-acyclic cofibration. Given any n, we may assume that f is a levelwise n-cofibration by Proposition 5.9. Since tensors with finite simplicial sets can be constructed levelwise and since pushouts can be constructed levelwise [AM, App. 4.2], it follows from Proposition 4.11 that g is a levelwise n-cofibration. This means that g is a levelwise n-equivalence for every n, so g is a π_*-weak equivalence.

For right properness, consider a pullback square

$$
\begin{array}{ccc}
W & \xrightarrow{q} & X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{p} & Z
\end{array}
$$

in which f is a π_*-weak equivalence and p is a π_*-fibration. We want to show that g is also a π_*-weak equivalence. Lemma 5.10 implies that p is a strict fibration. Therefore, the proof of [I2, Thm. 4.13] can be applied to show that base changes of essentially levelwise n-equivalences along π_*-fibrations are again essentially levelwise n-equivalences. We need Lemma 4.4 for the proof to work.

The proof of left properness is dual.   □

Remark 5.17. The proof of properness for pro-spaces given in [I1, Prop. 17.1] is incorrect, but the techniques of [I2, Thm. 4.13] can be used to fix it.

We write $\operatorname{Map}(X, Y)$ for the simplicial mapping space of pro-maps from X to Y. More precisely, we have the formula

$$\operatorname{Map}(X, Y) = \lim_s \operatorname{colim}_t \operatorname{Map}(X_t, Y_s).$$

Constructing cofibrant replacements is straightforward. Given a pro-spectrum X, we just take a levelwise cofibrant replacement. In Section 6, we will show that

constructing π_*-fibrant replacements is a bit more complicated. Let X be a pro-spectrum indexed by a cofiltered category I. Define a new pro-spectrum PX indexed by $I \times \mathbb{Z}$ as follows. For every pair (s, n), let $PX_{(s,n)} = P_n X_s$ be the nth Postnikov section of X_s. Finally, take a strict fibrant replacement for PX. The resulting pro-spectrum is a π_*-fibrant replacement for X.

5.6. **Stable model structure.** Recall that the functors $- \wedge S^1$ and $\mathrm{Map}(S^1, -)$ are defined levelwise for pro-spectra. In this section, we will show that the π_*-model structure is stable in the sense that these functors are a Quillen equivalence from the π_*-model structure to itself.

Lemma 5.18. *The functors $- \wedge S^1$ and $\mathrm{Map}(S^1, -)$ are a Quillen adjoint pair from the π_*-model structure on pro-spectra to itself.*

Proof. On spectra, $- \wedge S^1$ preserves cofibrations. Therefore, it preserves levelwise cofibrations and thus essentially levelwise cofibrations.

On spectra, $- \wedge S^1$ takes n-cofibrations to $(n+1)$-cofibrations. With the help of Proposition 5.9, this shows that $- \wedge S^1$ preserves π_*-acyclic cofibrations. $\square$

Lemma 5.19. *Let A and B be cofibrant pro-spectra. A map $f : A \to B$ is a π_*-weak equivalence if and only if $f \wedge S^1$ is a π_*-weak equivalence.*

Proof. To simplify notation, write F for the functor $- \wedge S^1$ and G for the functor $\mathrm{Map}(S^1, -)$.

One direction follows immediately from Lemma 5.18 and the fact that left Quillen functors preserve weak equivalences between cofibrant objects [Hi, Prop. 8.5.7].

For the other direction, suppose that Ff is a π_*-weak equivalence. Factor f into a cofibration $i : A \to C$ followed by a strict acyclic fibration $p : C \to B$. The map Fp is a strict weak equivalence because left Quillen functors preserve weak equivalences between cofibrant objects. By the two-out-of-three axiom, Fi is also a π_*-weak equivalence. By the two-out-of-three axiom again, f is a π_*-weak equivalence if i is a π_*-weak equivalence. Therefore, it suffices to show that the cofibration i is a π_*-weak equivalence.

We use the lifting characterization of Lemma 5.14 to show that i is a π_*-acyclic cofibration. We may assume that A and C are both levelwise cofibrant. Using the level replacement of [AM, App. 3.2], we may reindex A and C in such a way that i is a level map and A and C are still levelwise cofibrant. However, we are not allowed to assume that i is a levelwise cofibration because this may require a different reindexing.

Suppose given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & cX \\ \downarrow & & \downarrow \\ C & \longrightarrow & cY, \end{array}$$

where $X \to Y$ is a co-m-fibration for some m. This diagram of pro-spectra is represented by a diagram

$$\begin{array}{ccc} A_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ C_s & \longrightarrow & Y \end{array}$$

of spectra for some s. First factor, $C_s \to Y$ into a cofibration $C_s \to \tilde{Y}$ followed by an acyclic fibration $\tilde{Y} \to Y$. Then factor the map $A_s \to \tilde{Y} \times_Y X$ into a cofibration $A_s \to \tilde{X}$ followed by an acyclic fibration $\tilde{X} \to \tilde{Y} \times_Y X$. This gives us a diagram

$$
\begin{array}{ccccc}
A_s & \longrightarrow & \tilde{X} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
C_s & \longrightarrow & \tilde{Y} & \longrightarrow & Y
\end{array}
$$

in which the map $\tilde{X} \to \tilde{Y}$ is a fibration, the maps $\tilde{X} \to X$ and $\tilde{Y} \to Y$ are weak equivalences, and the spectra $\tilde{X}$ and $\tilde{Y}$ are cofibrant. By Proposition 4.8, $\tilde{X} \to \tilde{Y}$ is also a co-m-fibration. Now we have a diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & c\tilde{X} & \longrightarrow & cX \\
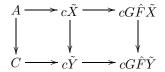\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & c\tilde{Y} & \longrightarrow & cY
\end{array}
$$

of pro-spectra. We want to show that the outer rectangle has a lift, so it suffices to show that the left square has a lift.

Let $\hat{F}\tilde{Y}$ be a fibrant replacement for $F\tilde{Y}$. Factor the composition $F\tilde{X} \to \hat{F}\tilde{Y}$ into an acyclic cofibration $F\tilde{X} \to \hat{F}\tilde{X}$ followed by a fibration $\hat{F}\tilde{X} \to \hat{F}\tilde{Y}$. Note that $\hat{F}\tilde{X}$ is a fibrant replacement for $F\tilde{X}$.

The maps $\tilde{X} \to G\hat{F}\tilde{X}$ and $\tilde{Y} \to G\hat{F}\tilde{Y}$ are weak equivalences because F and G are a Quillen equivalence on spectra. Here is where we use that $\tilde{X}$ and $\tilde{Y}$ are cofibrant. Because G is a right Quillen functor, the map $G\hat{F}\tilde{X} \to G\hat{F}\tilde{Y}$ is also a fibration. Moreover, this fibration is a co-m-fibration by Proposition 4.8.

Now consider the diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & c\tilde{X} & \longrightarrow & cG\hat{F}\tilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & c\tilde{Y} & \longrightarrow & cG\hat{F}\tilde{Y}
\end{array}
$$

of pro-spectra. A lift exists in the outer rectangle by adjointness and Lemma 5.14 applied to the π_*-acyclic cofibration $Fi : FA \to FC$. By [CDI, Prop. 3.2], a lift exists in the left square also. $\qquad\square$

Theorem 5.20. *The functors* $-\wedge S^1$ *and* $\mathrm{Map}(S^1, -)$ *are a Quillen equivalence from the* π_*-*model structure on pro-spectra to itself.*

Proof. As before, to simplify the notation, write F for $-\wedge S^1$ and G for $\mathrm{Map}(S^1, -)$.

Suppose that $g : X \to GY$ is any map such that X is cofibrant and Y is π_*-fibrant. We want to show that g is a π_*-weak equivalence if and only if its adjoint $f : FX \to Y$ is a π_*-weak equivalence.

Factor g into a cofibration $i : X \to Z$ followed by a strict acyclic fibration $p : Z \to GY$. The adjoint $p' : FZ \to Y$ is a strict weak equivalence because F and G are a Quillen equivalence on the strict model structure as shown in Theorem 3.6. Here we are using that Y is strict fibrant by Lemma 5.10.

The adjoint f is the composition of Fi with p'. By the two-out-of-three axiom, f is a π_*-weak equivalence if and only if Li is a π_*-weak equivalence. Because X and Z are cofibrant, Lemma 5.19 tells us that Fi is a π_*-weak equivalence if and only

if i is a π_*-weak equivalence. Finally, the two-out-of-three axiom implies that i is a π_*-weak equivalence if and only if g is a π_*-weak equivalence. $\square$

6. π_*-FIBRANT PRO-SPECTRA

Theorem 6.1. *A pro-spectrum X is π_*-fibrant if and only if it is strict fibrant and essentially levelwise fibrant and bounded above.*

That X is levelwise fibrant and bounded above means that each X_s is fibrant and bounded above (see Definition 4.2); we require no uniformity on the dimension in which the homotopy groups vanish.

The following proof is similar to the proof of [CI, Prop. 4.9].

Proof. First suppose that X is strict fibrant and essentially levelwise fibrant and bounded above. We will show that for every π_*-acyclic cofibration $i : A \to B$, the map $f : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ of simplicial sets is an acyclic fibration. By the usual adjointness arguments, this will show that $X \to *$ has the desired lifting property. Since X is strict fibrant and i is a cofibration, we already know that f is a fibration. It remains to show that f is a weak equivalence.

We may assume that each X_s is fibrant and bounded above. We showed in Lemma 5.12 that the constant pro-spectrum cX_s is π_*-fibrant. Therefore the map $\mathrm{Map}(B, cX_s) \to \mathrm{Map}(A, cX_s)$ is an acyclic fibration and in particular a weak equivalence.

Since X is levelwise fibrant and strict fibrant, Proposition 3.4 implies that the mapping space $\mathrm{Map}(A, X)$ is weakly equivalent to $\mathrm{holim}_s \mathrm{Map}(A, cX_s)$ (and similarly for $\mathrm{Map}(B, X)$). Homotopy limits preserve weak equivalences, so we conclude that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence. This completes one implication.

Now suppose that X is π_*-fibrant. Then X is strict fibrant by Lemma 5.10. It remains to show that X is essentially levelwise fibrant and bounded above.

Consider the factorization $X \to Y \to *$ of the map $X \to *$ into a π_*-acyclic cofibration followed by a π_*-fibration by means of the generalized small object argument (see Section 5). Now X is a retract of Y because X is π_*-fibrant and $X \to Y$ is a π_*-acyclic cofibration. The class of pro-objects having any property essentially levelwise is closed under retracts [I3, Thm. 5.5], so it suffices to consider Y.

Recall that $Y \to *$ is constructed as a composition of a transfinite tower

$$\cdots \to Y_\beta \to \cdots \to Y_2 \to Y_1 \to *,$$

where each map $Y_{\beta+1} \to Y_\beta$ is a base change of a product of maps of the form $cE \to cB$ with $E \to B$ a co-m-fibration for some m. The class of pro-objects having any property essentially levelwise is closed under cofiltered limits [I3, Thm. 5.1], so it suffices to consider each Y_β.

We proceed by transfinite induction. When β is a limit ordinal, [I3, Thm. 5.1] again tells us that Y_β is essentially levelwise fibrant and bounded above.

It only remains to consider the case when β is a successor ordinal. We are assuming that $Y_{\beta-1}$ is levelwise fibrant and bounded above. We may take a level representation for the diagram

$$Y_{\beta-1} \longrightarrow \prod_a cB_a \longleftarrow \prod_a cE_a,$$

where each $E_a \to B_a$ is a co-m-fibration for some m. Note that m depends on a. We construct Y_β by taking the levelwise fiber product. It is possible to construct a level representation for the above diagram in such a way that the replacement for $Y_{\beta-1}$

is a diagram of objects that already appeared in the original $Y_{\beta-1}$. This means that the new $Y_{\beta-1}$ is still levelwise fibrant and bounded above.

The construction of arbitrary products in pro-categories [I1, Prop. 11.1] shows that the map $\prod_a cE_a \to \prod_a cB_a$ is levelwise a finite product of maps of the form $E_a \to B_a$. A finite product of maps that are co-m-fibrations for some m is again a co-m-fibration for some m, so the map $Y_\beta \to Y_{\beta-1}$ is levelwise a base change of a co-m-fibration for some m. Since co-m-fibrations are closed under base change, we conclude that $Y_\beta \to Y_{\beta-1}$ is a levelwise co-m-fibration. It follows immediately that Y_β is levelwise fibrant and bounded above. $\square$

Remark 6.2. Similarly to [I1, Prop. 6.6] and [I2, Defn. 4.2], it is possible to give a concrete description of the π_*-fibrations. Recall that a directed set is cofinite if for every s, there are only finitely many t such that $t \leq s$. Suppose that $f : X \to Y$ is a level map indexed by a cofinite directed set such that each map

$$X_s \to Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} X_t$$

is a co-m-fibration for some m. Here m depends on s. Then f is a π_*-fibration. Up to retract, every π_*-fibration is of this form.

The following corollary simplifies the construction of π_*-fibrant replacements.

Corollary 6.3. *If Y is an essentially levelwise bounded above pro-spectrum, then there is a strict fibrant replacement $\hat{Y}$ for Y such that $\hat{Y}$ is also a π_*-fibrant replacement for Y.*

Proof. We may assume that Y is levelwise bounded above. Factor the map $Y \to *$ into a strict acyclic cofibration $Y \to \hat{Y}$ followed by a strict fibration $\hat{Y} \to *$ using the method of [I2, Lem. 4.7]. This particular construction gives that $Y \to \hat{Y}$ is a levelwise weak equivalence and $\hat{Y} \to *$ is a levelwise fibration; thus $\hat{Y}$ is levelwise fibrant and bounded above. Now Theorem 6.1 implies that $\hat{Y}$ is π_*-fibrant. $\square$

The next corollary simplifies the computation of mapping spaces of pro-spectra.

Corollary 6.4. *Let X be a cofibrant pro-spectrum, and let Y be a levelwise fibrant bounded above pro-spectrum with π_*-fibrant replacement $\hat{Y}$. Then the homotopically correct mapping space $\mathrm{Map}(X, \hat{Y})$ is weakly equivalent to $\mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$.*

Proof. Because the mapping space is homotopically correct, it doesn't matter which π_*-fibrant replacement $\hat{Y}$ we consider. Thus, we may take the one from Corollary 6.3. Because $\hat{Y}$ is a strict fibrant replacement for Y, Proposition 3.4 can be applied. $\square$

7. HOMOTOPY CLASSES OF MAPS OF PRO-SPECTRA

Let $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathbf{pro}}$ be the set of weak homotopy classes from X to Y in the π_*-homotopy category of pro-spectra. Let $[\boldsymbol{X}, \boldsymbol{Y}]^{\boldsymbol{r}}_{\mathbf{pro}}$ be the set of weak homotopy classes of degree r from X to Y. For all r, $[X, Y]^r_{\mathrm{pro}}$ is equal to

$$[\Sigma^{-r} X, Y]_{\mathrm{pro}} = [X, \Sigma^r Y]_{\mathrm{pro}},$$

where Σ^r equals Ω^{-r} if $r < 0$.

The mapping space $\mathrm{Map}(X, Y)$ is related to homotopy classes in the following way. For every cofibrant X, fibrant Y, and $r \geq 0$,

$$[X, Y]^{-r}_{\mathrm{pro}} \cong \pi_r \mathrm{Map}(X, Y) \cong \pi_0 \mathrm{Map}(\Sigma^r X, Y).$$

Proposition 7.1. *Let X be a pro-spectrum and Y be a bounded above spectrum. Then $[X, cY]^r_{\mathrm{pro}}$ is equal to $\mathrm{colim}_s[X_s, Y]^r$.*

Proof. We may assume that X is levelwise cofibrant and that $\Sigma^r Y$ is a fibrant spectrum. We must calculate homotopy classes of maps from X to $c\Sigma^r Y$. Now $\Sigma^r Y$ is bounded above since its homotopy groups are just the shifted homotopy groups of Y. Thus Theorem 6.1 tells us that the constant pro-spectrum $c\Sigma^r Y$ is already π_*-fibrant. Therefore,

$$[X, cY]^r_{\mathrm{pro}} \cong \pi_0 \mathrm{Map}(X, c\Sigma^r Y) \cong \mathrm{colim}_s \pi_0 \mathrm{Map}(X_s, \Sigma^r Y) \cong \mathrm{colim}_s [X_s, Y]^r.$$

$\square$

Proposition 7.1 is certainly false if Y is not bounded above. For example, let X be the pro-spectrum from Section 2, and let Y be the spectrum KU. Since X is contractible, $[X, cKU]_{\mathrm{pro}}$ is zero. On the other hand, we showed in Section 2 that $\mathrm{colim}_s[X_s, KU]$ is uncountable.

Lemma 7.2. *Let $* \to X$ be an essentially levelwise n-cofibration, and let $Y \to *$ be an essentially levelwise co-n-fibration. Then the homotopically correct mapping space $\mathrm{Map}(X, \hat{Y})_{\mathrm{pro}}$ is trivial, where $\hat{Y}$ is a π_*-fibrant replacement.*

Proof. We may assume that $* \to X$ is a levelwise n-cofibration and that $Y \to *$ is a levelwise co-n-fibration. In particular, this implies that Y is levelwise bounded above.

By Corollary 4.12, each space $\mathrm{Map}(X_t, Y_s)$ is contractible. Therefore, the filtered colimit $\mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$ is also contractible. It follows that the cofiltered homotopy limit $\mathrm{holim}_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$ is still contractible. Finally, Corollary 6.4 implies that this homotopy limit is weakly equivalent to $\mathrm{Map}(X, \hat{Y})$. $\square$

Corollary 7.3. *Let $* \to X$ be an essentially levelwise n-equivalence, and let $Y \to *$ be an essentially levelwise co-n-equivalence. Then $[X, Y]_{\mathrm{pro}}$ is zero.*

Proof. We may assume that X is cofibrant, so Proposition 5.9 implies that we may assume that $* \to X$ is a levelwise n-cofibration.

We may assume that that $Y \to *$ is a levelwise co-n-equivalence. By taking a levelwise fibrant replacement, we may further assume that $Y \to *$ is a levelwise fibration; thus $Y \to *$ is a levelwise co-n-fibration by Proposition 4.8.

Now the hypotheses of Lemma 7.2 are satisfied, so the homotopically correct mapping space is contractible. This implies that $[X, Y]_{\mathrm{pro}}$ is trivial. $\square$

8. Pro-homotopy groups

In this section, we give an alternative characterization of the π_*-weak equivalences of Definition 5.1. First we must discuss the stable homotopy pro-groups of a pro-spectrum. Since π_k is a functor on spectra, we may apply it objectwise to any pro-spectrum X to obtain a pro-group $\pi_k X$.

Proposition 8.1. *Let $i : A \to B$ be a cofibration with cofiber C. Then there is a long exact sequence*

$$\cdots \to \pi_k A \to \pi_k B \to \pi_k C \to \pi_{k-1} A \cdots$$

of pro-homotopy groups.

To understand what exactness means for this sequence, see [AM, App. 4.5] for a discussion of the abelian structure on the category of pro-abelian groups.

Proof. We may suppose that i is a level cofibration, and we may construct C as the levelwise cofiber of i because finite colimits in pro-categories can be constructed levelwise [AM, App. 4.2]. Now for every s, we have a long exact sequence

$$\cdots \to \pi_k A_s \to \pi_k B_s \to \pi_k C_s \to \pi_{k-1} A_s \to \cdots$$

of abelian groups. These sequences assemble to give the desired sequence. $\qquad\square$

Proposition 8.2. *Let $p : X \to Y$ be a π_*-fibration with fiber F. Then there is a long exact sequence*

$$\cdots \to \pi_k F \to \pi_k X \to \pi_k Y \to \pi_{k-1} F \cdots$$

of pro-homotopy groups.

Proof. By Lemma 5.11, we may assume that p is a levelwise fibration. We may construct F as the levelwise fiber of p because finite limits in pro-categories can be constructed levelwise [AM, App. 4.2]. Now for every s, we have a long exact sequence

$$\cdots \to \pi_k F_s \to \pi_k X_s \to \pi_k Y_s \to \pi_{k-1} F_s \to \cdots$$

of abelian groups. These sequences assemble to give the desired sequence. $\qquad\square$

Proposition 8.3. *Suppose that $j : A \to B$ is a cofibration of pro-spectra such that $\pi_k j$ is an isomorphism of pro-groups for every k and such that j is an essentially levelwise n-equivalence for some n. Then j is a π_*-acyclic cofibration.*
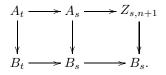
Proof. We may assume that j is a levelwise n-cofibration because of Proposition 5.9. We will show that j is a levelwise $(n+1)$-cofibration. By induction, this will imply that j is a levelwise m-cofibration for every m and hence a levelwise m-equivalence for every m by Proposition 4.9. This means that j is a π_*-weak equivalence.

For any s, we have a map $j_s : A_s \to B_s$. Factor j_s into an $(n+1)$-cofibration $i_{s,n+1} : A_s \to Z_{s,n+1}$ followed by a co-$(n+1)$-fibration $p_{s,n+1} : Z_{s,n+1} \to B_s$ as in Definition 5.13.

Let C be the cofiber of j, which we may assume is constructed levelwise. From the long exact sequence of Proposition 8.1, we see that $\pi_n C$ is the trivial pro-group. Therefore, we may choose $t \geq s$ such that the map $\pi_{n+1} C_t \to \pi_{n+1} C_s$ is zero. Note also that the map $* \to C_t$ is an n-equivalence because the map $A_t \to B_t$ is an n-cofibration.

Let F be the fiber of $p_{s,n+1}$. Note that the map $F \to *$ is a co-$(n+1)$-equivalence because $p_{s,n+1}$ is a co-$(n+1)$-fibration.
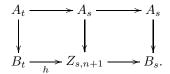
Consider the diagram

$$
\begin{array}{ccccc}
A_t & \longrightarrow & A_s & \longrightarrow & Z_{s,n+1} \\
\downarrow & & \downarrow & & \downarrow \\
B_t & \longrightarrow & B_s & \longrightarrow & B_s.
\end{array}
$$

The obstruction to lifting the right square is an element α of $[\Omega C_s, F]$, and the obstruction to lifting the outer rectangle is the image of α under the map $[\Omega C_s, F] \to [\Omega C_t, F]$ [CDI, Cor. 8.4]. We have chosen t such that the map $\pi_n \Omega C_t \to \pi_n \Omega C_s$ is zero. Also, note that $* \to \Omega C_t$ is an $(n-1)$-equivalence because its suspension $* \to C_t$ is an n-equivalence. Therefore, the conditions of Lemma 4.13 apply, and we

conclude that the map $[\Omega C_s, F] \to [\Omega C_t, F]$ is zero. Thus, the obstruction for lifting the outer square, which lies in the image of this map, must be zero, and a lift h exists.

Using this lift h, we get a diagram

$$
\begin{array}{ccccc}
A_t & \longrightarrow & A_s & \longrightarrow & A_s \\
\downarrow & & \downarrow & & \downarrow \\
B_t & \xrightarrow{\ h\ } & Z_{s,n+1} & \longrightarrow & B_s.
\end{array}
$$

Now the conditions of Lemma 3.2 are satisfied, so the objects $Z_{s,n+1}$ assemble into a pro-spectrum that is isomorphic to B. The maps $A_s \to Z_{s,n+1}$ are thus a level representation for j; this demonstrates that j is an essentially levelwise $(n + 1)$-equivalence. $\qquad\square$

Theorem 8.4. *A map of pro-spectra f is a π_*-weak equivalence (see Definition 5.1) if and only if $\pi_k f$ is an isomorphism of pro-abelian groups for every k and f is an essentially levelwise n-equivalence for some n.*

The second condition in the above theorem feels unnatural. It sounds plausible to construct a model structure on pro-spectra in which the weak equivalences are just pro-homotopy group isomorphisms, but we have no idea how to do this. One way to rationalize the existence of the second condition is that the cofibrations and fibrations are both plausible, and this leaves no choice in what the weak equivalences are.

Proof. First suppose that f is a π_*-weak equivalence, so f is an essentially levelwise n-equivalence for every n. For any k, choose a level representation for f that is a levelwise $(k + 1)$-equivalence. Then $\pi_k f$ is a levelwise isomorphism, so it is an isomorphism of pro-groups. This finishes one implication.

For the other implication, suppose that $\pi_k f$ is an isomorphism of pro-abelian groups for every k and f is an essentially levelwise n-equivalence for some n. Factor f into a cofibration i followed by a strict acyclic fibration p. Therefore, p is a strict weak equivalence. This means that $\pi_k p$ is an essentially levelwise isomorphism, so it is an isomorphism of pro-groups. We can conclude that $\pi_k i$ is an isomorphism of pro-groups.

Also note that i is an essentially levelwise $(n - 1)$-equivalence by Lemma 5.6. Therefore, Proposition 8.3 applies, and we can conclude that i is a π_*-weak equivalence. Since p is also a π_*-weak equivalence, the two-out-of-three axiom tells us that f is a π_*-weak equivalence. $\qquad\square$

9. COHOMOLOGY AND THE WHITEHEAD THEOREM

One of the primary motivations for the construction of our model structure is the study of cohomology of pro-spectra. We now explore the relationship between π_*-weak equivalences and cohomology isomorphisms. We recall first the definition of cohomology for pro-spectra.

Let HA be a fibrant Eilenberg-Mac Lane spectrum such that $\pi_0 HA = A$, where A is an abelian group.

Definition 9.1. The **rth cohomology $H^r(X; A)$ with coefficients in A** of a pro-spectrum X is the abelian group $[X, cHA]_{\mathrm{pro}}^{-r}$.

This definition is precisely analogous to the definition of ordinary cohomology for spectra.

Proposition 9.2. *If X is any pro-spectrum, then the cohomology group $H^r(X; A)$ is isomorphic to $\operatorname{colim}_s H^r(X_s; A)$.*

Proof. The spectrum HA has no homotopy groups above dimension 0, so Proposition 7.1 applies. $\square$

The previous proposition shows that our definition of ordinary cohomology in terms of Eilenberg-Mac Lane constant pro-spectra agrees with the traditional notion of the cohomology of a pro-object. Our viewpoint is that the straightforward colimit formula for cohomology works because HA is bounded above and therefore cHA is π_*-fibrant.

We now work toward a Whitehead theorem for detecting π_*-weak equivalences in terms of cohomology.

Lemma 9.3. *Let $i : A \to B$ be a cofibration that is an ordinary cohomology isomorphism for all coefficients, and let $X \to Y$ be a fibration of spectra whose fiber F is an Eilenberg-Mac Lane spectrum. Then i has the left lifting property with respect to the constant map $q : cX \to cY$.*

Proof. We may assume that i is a levelwise cofibration. Let C be the cofiber of i, which we may assume is constructed levelwise. The long exact sequence in cohomology for a cofiber sequence indicates that the cohomology of C is zero. Let k be the integer such that $\pi_k F$ is non-zero.

Consider a square

$$
\begin{array}{ccc}
A & \longrightarrow & cX \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle q} \\
B & \longrightarrow & cY
\end{array}
$$

of pro-spectra. This diagram is represented by a square

$$
\begin{array}{ccc}
A_s & \longrightarrow & X \\
\downarrow{\scriptstyle i_s} & & \downarrow{\scriptstyle q} \\
B_s & \longrightarrow & Y
\end{array}
$$

of spectra. The obstruction α to lifting this square is a weak homotopy class belonging to $[\Omega C_s, F]$ [CDI, Rem. 8.3].

Note that $[\Omega C_s, F]$ equals $H^{k+1}(C_s; \pi_k F)$ because F is an Eilenberg-Mac Lane spectrum. Because $H^{k+1}(C; \pi_k F)$ is zero, there exists a t such that α pulls back to zero in $H^{k+1}(C_t; \pi_k F)$. Therefore, the obstruction to lifting the square

$$
\begin{array}{ccccc}
A_t & \longrightarrow & A_s & \longrightarrow & X \\
\downarrow{\scriptstyle i_t} & & & & \downarrow{\scriptstyle q} \\
B_t & \longrightarrow & B_s & \longrightarrow & Y
\end{array}
$$

vanishes, and a lift exists. This lift represents the desired lift. $\square$

Theorem 9.4. *Let $f : X \to Y$ be an essentially levelwise n-equivalence for some n. Then f is a π_*-weak equivalence if and only if it is an ordinary cohomology isomorphism for all coefficients.*

Proof. One direction is easy; since cohomology is represented in the π_*-homotopy category, π_*-weak equivalences are cohomology isomorphisms.

For the other direction, let $f : X \to Y$ be an essentially levelwise n-equivalence for some n and an ordinary cohomology isomorphism for all coefficients. Factor f into a cofibration $i : X \to Z$ followed by a π_*-acyclic fibration $p : Z \to Y$. Then i is still an essentially levelwise n-equivalence for some n by Lemma 5.6. Also, p is a π_*-weak equivalence, so it is a cohomology isomorphism since cohomology is defined to be representable. This means that i is a cohomology isomorphism, and we just have to show that i is a π_*-acyclic cofibration.

We may assume that i is a levelwise n-cofibration by Proposition 5.9. We use Lemma 5.14 to show that i is a π_*-acyclic cofibration. Thus, we must find a lift in the square

$$
\begin{array}{ccc}
X & \longrightarrow & cE \\
{\scriptstyle i}\big\downarrow & & \big\downarrow \\
Z & \longrightarrow & cB
\end{array}
$$

of pro-spectra, where $q : E \to B$ is a co-m-fibration for some m. This diagram is represented by a square

$$
\begin{array}{ccc}
X_s & \longrightarrow & E \\
{\scriptstyle i_s}\big\downarrow & & \big\downarrow \\
Z_s & \longrightarrow & B
\end{array}
$$

of spectra, and we have to find a lift after refining s. According to Lemma 4.10, q is a retract of a finite composition of maps that are either co-n-fibrations or fibrations whose fiber is an Eilenberg-Mac Lane spectrum. Since we are trying to solve a lifting problem, we may assume that q is a co-n-fibration or has an Eilenberg-Mac Lane spectrum as its fiber.

If q is a co-n-fibration, then a lift exists without refining s at all since i_s is an n-cofibration. In the other case, Lemma 9.3 produces the lift. $\square$

10. Atiyah-Hirzebruch Spectral Sequence for Pro-Spectra

We now consider generalized cohomology for pro-spectra.

Definition 10.1. Let E be any fixed pro-spectrum. The **rth E-cohomology $E^r(X)$** of a pro-spectrum X is the abelian group $[X, E]_{\mathrm{pro}}^{-r}$.

This definition is precisely analogous to the definition of generalized cohomology for spectra. In general, the calculation of $E^* X$ requires a fibrant replacement of the pro-spectrum E. When E is constant, the Postnikov tower of E is one possible such fibrant replacement.

For example, if E is the constant pro-spectrum cKU, then $KU^r(X)$ is equal to $[X, P_* KU]_{\mathrm{pro}}$, where $P_* KU$ is the Postnikov tower of KU. Since KU is not bounded above, it is not true that $KU^r(X)$ is equal to $\mathrm{colim}_s KU^r(X_s)$; the hypothesis of Proposition 7.1 is not satisfied.

We now develop an analogue of the Atiyah-Hirzebruch spectral sequence [AH] for pro-spectra. Here we are addressing the question of computing $[X, Y]_{\mathrm{pro}}^{-r}$ for an arbitrary pro-spectrum X and an arbitrary pro-spectrum Y.

We fix a pro-spectrum Y. Let A^q be the pro-spectrum $P_{-q}Y$, i.e., the levelwise $(-q)$th Postnikov section of Y. We choose the unusual indexing on A^q in order to standardize the indexing of our cohomological spectral sequence. There is a diagram

$$\cdots \to A^{q-1} \to A^q \to A^{q+1} \to \cdots,$$

and we let A be the inverse limit (in the category of pro-spectra) of this tower. As a cofiltered diagram, A is described by $(s,q) \mapsto P_{-q}Y_s$. Note that a strict fibrant replacement for A is a π_*-fibrant replacement for Y.

For any spectrum Z, let C_qZ be the qth connected cover of Z, i.e., the homotopy fiber of the map $Z \to P_qZ$. Let B^q be the pro-spectrum $C_{-q}Y$, i.e., the levelwise $(-q)$-connected cover of Y. Again, there is a diagram

$$\cdots \to B^{q-1} \to B^q \to B^{q+1} \to \cdots,$$

and we let B be the inverse limit (in the category of pro-spectra) of this tower. As a cofiltered diagram, B is described by $(s,q) \mapsto C_{-q}Y_s$.

We next show that B is contractible in the π_*-model structure.

Lemma 10.2. *The map $* \to B$ is a π_*-weak equivalence.*

Proof. Fix an integer n. The pro-spectrum $B = \lim_q B^q$ is isomorphic to the pro-spectrum $\lim_{q<-n} B^q$. Every object of $\lim_{q<-n} B^q$ is of the form $C_{-q}Y_s$ for some s and some $q < -n$. The map $* \to C_{-q}Y_s$ is an n-equivalence because $-q > n$. Thus, $* \to \lim_{q<-n} B^q$ is a levelwise n-equivalence, so $* \to B$ is an essentially levelwise n-equivalence. Since n was arbitrary, this shows that $* \to B$ is a π_*-weak equivalence. $\square$

Recall that for every s, $\Sigma^{-q}H\pi_{-q}Y_s$ is an Eilenberg-Mac Lane spectrum whose only non-zero homotopy group lies in dimension $-q$ and is isomorphic to $\pi_{-q}Y_s$. Let $\Sigma^{-q}H\pi_{-q}Y$ be the obvious pro-spectrum constructed out of these Eilenberg-Mac Lane spectra.
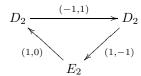
Lemma 10.3. *For every q, the sequence*

$$B^q \to B^{q+1} \to \Sigma^{-q}H\pi_{-q}Y$$

is a homotopy cofiber sequence of pro-spectra.

Proof. In order to compute the homotopy cofiber of any map, we should replace it by a levelwise cofibration and then take the cofiber, i.e., the levelwise cofiber. In other words, we just need to take the levelwise homotopy cofiber.

Recall that $B^q \to B^{q+1}$ is given levelwise by maps $C_{-q}Y_s \to C_{-q-1}Y_s$. The homotopy cofiber of $C_{-q}Y_s \to C_{-q-1}Y_s$ is $\Sigma^{-q}H\pi_{-q}Y_s$. $\square$

Let X be any pro-spectrum. Define $\boldsymbol{D_2^{p,q}}$ to be $[X, B^q]_{\mathrm{pro}}^{p+q}$, and define $\boldsymbol{E_2^{p,q}}$ to be $[X, \Sigma^{-q}H\pi_{-q}Y]_{\mathrm{pro}}^{p+q}$. The π_*-model structure is stable from Theorem 5.20, so the homotopy cofiber sequence of Lemma 10.3 is also a homotopy fiber sequence [Ho, Thm. 7.1.11]. After applying the functor $[X, -]_{\mathrm{pro}}$, one obtains a long exact sequence. Therefore, we have an exact couple

$$\begin{array}{ccc} D_2 & \xrightarrow{\ (-1,1)\ } & D_2 \\ {\scriptstyle (1,0)}\ \nwarrow & & \swarrow\ {\scriptstyle (1,-1)} \\ & E_2 & \end{array}$$

in which the labels indicate the degrees of the maps. A careful inspection of degrees shows that this gives us a spectral sequence beginning with the E_2-term.

Now we have a spectral sequence, but we must study its convergence. We take the viewpoint of [B].

Lemma 10.4. *For all n and all X, the groups $\lim_q D_2^{q,n-q}$ and $\lim_q^1 D_2^{q,n-q}$ vanish as $q \to \infty$.*

Proof. Let $\hat{B}^q$ be the pro-spectrum described by $(s,p) \mapsto P_{-p}C_{-q}Y_s$. The map $B^q \to \hat{B}^q$ is a π_*-weak equivalence and $\hat{B}^q$ is levelwise bounded above. By taking a levelwise fibrant replacement, we may additionally assume that $\hat{B}^q$ is levelwise fibrant.

Let $\hat{B}$ be the inverse limit $\lim_q \hat{B}^q$ (computed in the category of pro-spectra). As a cofiltered diagram, $\hat{B}$ is described by $(s,p,q) \mapsto P_{-p}C_{-q}Y_s$. Again, $\hat{B}$ is levelwise bounded above, and the map $B \to \hat{B}$ is a π_*-weak equivalence. Since each $\hat{B}^q$ is levelwise fibrant, so is $\hat{B}$.

We already know that $[X,B]_{\mathrm{pro}}^n$ is zero for all n because B is contractible. However, we will compute these homotopy classes another way by considering the components of the appropriate homotopically correct mapping space.

Take a cofibrant model for $\Sigma^{-n}X$. Corollary 6.4 says that the homotopically correct mapping space for computing maps from $\Sigma^{-n}X$ to B is weakly equivalent to $\operatorname{holim}_{s,p,q} \operatorname{colim}_t \operatorname{Map}(\Sigma^{-n}X_t, \hat{B}_{s,p}^q)$. Since homotopy limits commute, we can compute this as

$$\operatorname*{holim}_q \operatorname*{holim}_{s,p} \operatorname*{colim}_t \operatorname{Map}(\Sigma^{-n}X_t, \hat{B}_{s,p}^q).$$

Now for a fixed q,

$$M^q = \operatorname*{holim}_{s,p} \operatorname*{colim}_t \operatorname{Map}(\Sigma^{-n}X_t, \hat{B}_{s,p}^q)$$

is weakly equivalent again by Corollary 6.4 to the homotopically correct mapping space for computing maps from $\Sigma^{-n}X$ to B^q.

Now apply the short exact sequence of [BK, Thm.IX.3.1] for computing the homotopy groups of the homotopy limit of the countable tower

$$\cdots \to M^{q-1} \to M^q \to M^{q+1} \to \cdots .$$

We obtain the sequence

$$0 \to \lim_q^1 \pi_1 M^q \to \pi_0 \operatorname{holim}_q M^q \to \lim_q \pi_0 M^q \to 0.$$

We already know that the middle term of the sequence is zero because it is equal to $[X,B]_{\mathrm{pro}}^n$. Therefore, the first and last terms are also zero. The first term is $\lim_q^1 [X,B^q]_{\mathrm{pro}}^{n-1}$, and the last term is $\lim_q [X,B^q]^n$.

Since n is arbitrary, we have shown that as $q \to -\infty$, both $\lim_q [X,B^q]^n$ and $\lim_q^1 [X,B^q]^n$ are zero for all n. Changing indices gives that $\lim_q [X,B^{n-q}]^n$ and $\lim_q^1 [X,B^{n-q}]^n$ are both zero as $q \to \infty$. $\square$

Lemma 10.5. *Suppose that the map $* \to X$ is an essentially levelwise m-equivalence for some m. For all n, as $q \to -\infty$, $\operatorname{colim}_q D_2^{q,n-q}$ is isomorphic to $[X,Y]_{\mathrm{pro}}^n$.*

Proof. We want to show that the natural map

$$\operatorname*{colim}_q [X, B^{n-q}]_{\mathrm{pro}}^n \to [X,Y]_{\mathrm{pro}}^n$$

is an isomorphism. The sequence $B^{n-q} \to Y \to A^{n-q}$ is a levelwise homotopy cofiber sequence (given by $C_{q-n}Y_s \to Y_s \to P_{q-n}Y_s$ for each s), so it is a homotopy cofiber sequence of pro-spectra. Therefore, it is also a homotopy fiber sequence, so we get a long exact sequence after applying $[X, -]_{\mathrm{pro}}$. By the usual argument with long exact sequences and the exactness of filtered colimits, it suffices to show that $\mathrm{colim}_q[X, A^{n-q}]_{\mathrm{pro}}^n$ is zero for every n.

We may assume that the map $* \to X$ is a levelwise m-equivalence, so the map $* \to \Sigma^{-n}X$ is a levelwise $(m - n)$-equivalence.

Choose $q' = m - 1$, so $q' - n = m - n - 1$. Then the map $A^{n-q'} \to *$ is a levelwise co-$(m - n)$-equivalence because each object of $A^{n-q'}$ is an $(m - n - 1)$st Postnikov section. Now the hypotheses of Corollary 7.3 are satisfied, so $[\Sigma^{-n}X, A^{n-q'}]_{\mathrm{pro}}$ is zero. Therefore, the colimit $\mathrm{colim}_q[X, A^{n-q}]_{\mathrm{pro}}^n$ is zero. $\square$

Lemma 10.6. *Suppose that X is an essentially bounded below pro-spectrum and that Y is a constant pro-spectrum. For all n, as $q \to -\infty$, $\mathrm{colim}_q D_2^{q,n-q}$ is isomorphic to $[X, Y]_{\mathrm{pro}}^n$.*

The hypothesis means that X is isomorphic to a pro-spectrum X' such that every X'_s is bounded below, but the dimension at which the homotopy groups of X'_s vanish may depend on s. For example, the pro-spectra $\mathbb{R}P_{-\infty}^\infty$ and $\mathbb{C}P_{-\infty}^\infty$ [L] are essentially bounded below.

Proof. We want to show that the natural map

$$\mathrm{colim}_q[X, B^{n-q}]_{\mathrm{pro}}^n \to [X, Y]_{\mathrm{pro}}^n$$

is an isomorphism. As in the proof of Lemma 10.5, it suffices to show that the group $\mathrm{colim}_q[\Sigma^{-n}X, A^{n-q}]_{\mathrm{pro}}$ is zero for every n.

We may assume that X is levelwise bounded below; then $\Sigma^{-n}X$ is also levelwise bounded below.

By Proposition 7.1, we must show that $\mathrm{colim}_{q,t}[\Sigma^{-n}X_t, A^{n-q}]$ is zero; here is where we use that Y and therefore A^{n-q} is constant. Fix an index t. By the assumption on X, there exists m such that $* \to X_t$ is an m-equivalence. Then the map $* \to \Sigma^{-n}X_t$ is an $(m - n)$-equivalence.

Choose $q' = m$, so $q' - n = m - n$. Now the homotopy groups of $A^{n-q'} *$ vanish in dimensions greater than or equal to $m-n+1$ because $A^{n-q'}$ is an $(m-n)$th Postnikov section. The homotopy groups of $\Sigma^{-n}X_t$ vanish in dimensions less than or equal to $m - n$. These conditions on the homotopy groups guarantee that $[\Sigma^{-n}X_t, A^{n-q'}]$ is zero. This shows that the colimit is zero. $\square$

Theorem 10.7. *Let X and Y be any pro-spectra. There is a spectral sequence*

$$E_2^{p,q} = H^{-p}(X; \pi_{-q}Y)$$

converging to $[X, Y]_{\mathrm{pro}}^{p+q}$. The differentials have degree $(r, -r + 1)$. The spectral sequence is conditionally convergent if:

(1) X is essentially levelwise bounded below, or

(2) Y is a constant pro-spectrum and $* \to X$ is an essentially levelwise n-equivalence for some n.

Proof. The conditional convergence comes from Lemmas 10.4, 10.5, and 10.6. The identification of the E_2-term is given in Definition 9.1. $\square$

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